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CHARACTERISTIC DISTURBANCE FREQUENCY IN VERTICAL NATURAL CONVECTION FLOW

BENJAMIN GEBHART and R. MAHAJAN
Silbey School of Mechanical and Aerospace Engineering, Upson Hall, Cornell University, Ithaca, NY 14853, U.S.A.

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Abstract—Many experiments, as well as stability analysis, indicate that a vertical natural convection boundary-layer flow sharply filters disturbances for essentially a single frequency (characteristic frequency) as the disturbances are convected downstream. The analysis indicates that the characteristic frequency, $B^* = \beta^* G^{*1/2}$ or $B = \beta G^{1/3}$ downstream, for uniform flux or isothermal flows, respectively, is only a function of Prandtl number. The particular values are calculated from stability planes for numerous Prandtl numbers and for $Pr \to 0$ and for $Pr \to \infty$. The resulting curves of B^* or B vs Pr number closely correlate the existing experimental results for various fluids for different heating and ambient conditions.

NOMENCLATURE

f, disturbance frequency for isothermal flow;
k, thermal conductivity;
q'', source heat flux;
x, streamwise coordinate;
Gr_x, $g\beta_T \Delta T x^3/\nu^2$;
Gr_x^*, $g\beta_T q'' x^4/k\nu^2$;
Pr, Prandtl number.

Greek symbols

α_i, disturbance amplification rate;
β, dimensionless frequency for isothermal case;
β^*, dimensionless frequency for uniform flux case;
β_T, thermal expansion coefficient;
ΔT, local temperature difference;
ν, kinematic viscosity coefficient.

1. INTRODUCTION

CALCULATIONS based on linear stability theory and available experimental data have shown that natural convection flows favor a narrow band of disturbance frequency for amplification. Eckert and Soehngen [1] studied the stability of a laminar flow in air, using an interferometer. They concluded that a boundary layer, subject to natural disturbances filtered for fluctuations of certain frequencies and amplified these until turbulent waves were produced. Later, Holman, Gartrell and Soehngen [2] introduced artificial oscillations and noted that the boundary layer absorbs energy more readily at some frequencies than at others. A single frequency was found to produce maximum wave amplitude. Polymeropoulos and Gebhart [3] found similar behavior.

The first analytical study which predicted sharp frequency filtering was that of Dring and Gebhart [4]. Coupled Orr–Sommerfield disturbance equations, for laminar natural convection boundary-layer flow over a vertical surface with uniform heat flux, were solved numerically. For uniform heat flux or isothermal flows, the Grashof number parameter G^* or G, in terms of convectional Grashof number Gr_x^* or Gr_x, is defined as $5\sqrt[5]{(Gr_x^*/5)}$ or $4\sqrt[4]{(Gr_x/4)}$ respectively where $Gr_x^* = g\beta_T q'' x^4/k\nu^2$ and $Gr_x = g\beta_T x^3 \Delta T/\nu^2$. Here x is the distance along the vertical surface from the leading edge, q'' the uniform heat flux, k the thermal conductivity, ν the kinematic viscosity, β_T the coefficient of thermal expansion, g the acceleration due to gravity and ΔT the local temperature difference at x, between the surface and the ambient fluid.

The stability plane, in terms of non-dimensionalized frequency β^* and G^* presented contours of equal downstream amplitude, for a fluid of Prandtl number $(Pr) = 6{\cdot}7$, up to $G^* = 300$. These contours represent e^A = constant, where $A = -\int \alpha_i \, dG^*/4$ and α_i is the downstream spatial amplification rate. The integral is taken along a constant physical frequency (f) path downstream from the neutral curve. Note that the physical and the non-dimensionalized frequencies for uniform flux and isothermal flow conditions are related by $f = \nu G^{*3}/50\pi x^2 \beta^*$ and $f = \nu G^3/32\pi x^2 \beta$, respectively, where β is the generalized frequency for isothermal flow. Thus the constant physical frequency paths for the two conditions are $\beta^* G^{*1/2}$ and $\beta G^{1/3}$, respectively. These trajectories when plotted on the stability plane clearly indicated that a very narrow band of frequencies is highly favored for amplification.

Gebhart [5] analyzed existing experimental data concerning the frequency of the first-appearing highly amplified disturbances or first signs of transition. The results, from six experiments, were plotted on the stability plane. They lay generally in the region which would be reached by disturbances of frequencies predicted to amplify at high rates.

Hieber and Gebhart [6] extended the stability calculations to fluids with Pr varying between 0·01 and 100. In a later paper [7], they obtained asymptotic solutions for the limit $Pr \to \infty$. The G^* range of the stability plane was also extended to 1000 for some conditions. In particular, detailed stability planes, with spatial amplification characteristics and constant disturbance frequency paths, were presented for $Pr = 6{\cdot}7$ and 0·733. These correspond to water and air. Past experimental frequency data pertaining to the onset of transition or observation of amplified disturbances, when located on the respective stability planes, confirmed the earlier conclusions of Gebhart [5]. That is, the points, in β^* and G^*, fell very close to paths of greatest amplification. Subsequent work of Godaux and Gebhart [8] and Jaluria and Gebhart [9–11] have since added further strong confirmation.

We may conclude then that, as a disturbance is convected downstream, it is filtered for a predominant component, hereafter called the characteristic frequency. Essentially all the disturbance energy is concentrated at this frequency. See a recent review by Gebhart [12]. It is also certain from the four studies cited, that these disturbances lead directly to transition. This characteristic disturbance causes transition.

The value of the characteristic frequency is defined as $B^* = \beta^* G^{*1/2} = 2\pi f(g\beta_T q''/kv^2)^{-1/2}$ or $B = \beta G^{1/3} = 2\pi f(g\beta_T \Delta T/v^2)^{-2/3}$. This is essentially a single value for a given fluid downstream to large G^* or G. See, for example, Fig. 1. It is only a function of Prandtl number. Both $B^*(Pr)$ and $B(Pr)$ will be determined.

DETERMINATION OF B^* AND B

How these quantities are determined for any Prandtl number depends on the amount of spatial disturbance amplification information available. For $Pr = 0{\cdot}733$ and 6·7 these are the extensive stability planes of Hieber and Gebhart [6]. A single value of $B^* = \beta^* \sqrt[2]{G^*}$ does not pass exactly through the minimum G^* values of all constant A contours. See Fig. 1. Therefore, the selection of B^* required some judgment.

Based on experimental evidence for fluids of Prandtl number 6·7 and 0·733, Hieber and Gebhart [6] suggested that first appearance of natural oscillations and transition fall approximately in the range $A = 6$ or 10. Thus the path crossing the amplitude curve $A = 8$ at its lowest G^* is taken to determine B^* for these Prandtl numbers. The points, at $B^* = 0{\cdot}675$ and at 1·342, are shown on Fig. 2. They are also listed in Table 1.

The asymptotic analysis of Hieber and Gebhart [7] for large Prandtl number, in their Fig. 12, yields $Pr^{2/5}\beta^* = 0{\cdot}12$ and $Pr^{-3/10}G^* = 75$ at the $A = 8$ location. These values imply

$$B^* = \beta^* \sqrt[2]{G^*} = 1{\cdot}04 Pr^{-1/4}. \qquad (1)$$

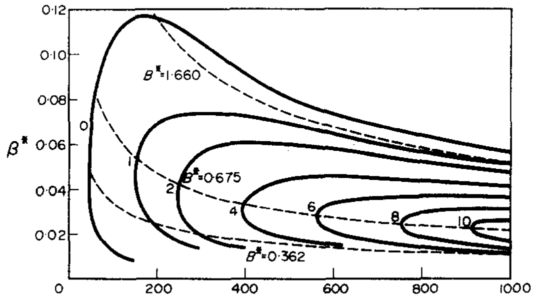

FIG. 1. Stability plane for $Pr = 6{\cdot}7$ (Hieber and Gebhart [6]) showing amplitude ratio contours in the unstable region. Dashed lines represent constant frequency paths.

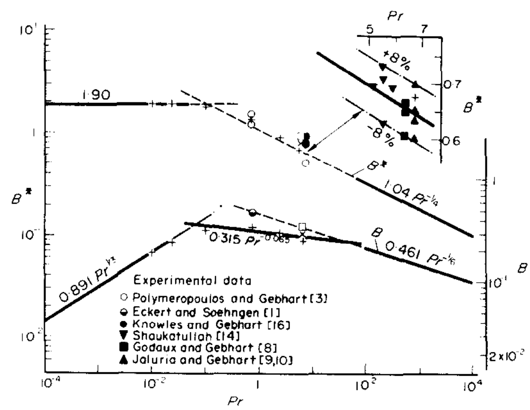

FIG. 2. Characteristic frequency for vertical natural convection flows; +, $J = 0$ (from [6, 7]); ×, $J = 1$, □, $J = 2$ (from [11]).

These results may be presumed to apply at least for $Pr > 100$, since the expansion parameter was $1/\sqrt{Pr}$. This result is also shown on Fig. 2 and is seen, remarkably, to also almost correlate the well-established points for $Pr = 0{\cdot}733$ and 6·7.

Although Hieber and Gebhart [6] also present amplification rate, α_i, contours for $Pr = 2{\cdot}5$, 0·1, 0·025 and 0·01, no A contours are given. We plotted constant frequency paths on these planes to determine the paths of most rapid amplification. The resulting conditions are also shown in Fig. 2 and listed in Table 1. For low Pr numbers, the points break away from the $Pr^{-1/4}$ trend, to a constant value.

Although no asymptotic solution to low Prandtl number was given in the above reference, the asymptotic behavior was investigated. It was found that, as $Pr \to 0$, the stability plane is to be in terms of coordinates $\beta^* Pr^{-1/5}$ and $G^* Pr^{2/5}$. Thus, any particular

Table 1. Characteristic frequency values based on stability theory

Pr	0·01	0·025	0·1	0·733	2·5	6·7		
J†	0	0	0	0	0	0	1	2
B^*	1·90	1·90	1·788	1·342	0·94	0·675	0·811	0·910
B	0·192	0·247	0·323	0·350	0·308	0·25	0·305	0·342

† $J = 4n(N_\infty/N)$ where $t_0 - t_\infty = Nx^n$ and $t_\infty - t_r = N_\infty x^n$; t_0, t_∞ and t_r being the surface, ambient and reference temperature respectively.

A location would occur at a $\beta^* \sqrt[3]{G^*} = B^*$ value independent of Prandtl number. This is in agreement with the points at $Pr = 0{\cdot}01$ and $0{\cdot}025$ on Fig. 2. An asymptotic value of 1·9 is chosen for low Prandtl number.

Thus we may conclude, for the uniform flux surface condition, that the two asymptotic trends accurately cover the whole range to within a few percent, the largest difference being about 20 per cent for $Pr = 0{\cdot}733$. For $Pr = 0{\cdot}1$, 2·5 and 6·7, the differences are about 3, 12 and 5 per cent respectively.

We note that all of the amplification results used above are based on the disturbance temperature boundary condition $s(0) = 0$. That is, the disturbance amplitude was assumed damped to zero at the surface by surface material thermal capacity. Calculations had shown that changing the condition to $s'(0) = 0$, a zero relative thermal capacity surface, produced essentially the same amplification results for $G^* \geqslant 500$, for high to moderate values of the Prandtl number. For low Prandtl number, however, amplification characteristics are somewhat different and, although a horizontal asymptote is still obtained, its level may be somewhat different. However, we do not know that the points for $Pr = 0{\cdot}01$ and $0{\cdot}025$ would lie at the same level.

The above results were calculated with the laminar base flow arising from a uniform surface flux condition. We may obtain estimates of characteristic frequency for an isothermal surface condition from the foregoing information by using the relation between the flux, q'', and the local heat-transfer temperature difference, $t_0 - t_\infty$. Thus local values of β^* and G^* may be converted to β and G, in terms of $\phi'(0)$, the temperature profile slope at the surface. This quantity, a function of the Prandtl number, is known from the base flow solution. The procedure is shown in Gebhart [13]. The result is

$$B = \beta \sqrt[3]{G} = 1{\cdot}03[-\phi'(0)^{8/15}]B^* G^{*-1/6}. \qquad (2)$$

For the two limiting cases, $Pr \to \infty$ and 0, the values of $[-\phi'(0)]$ are $0{\cdot}7962 Pr^{1/4}$ and $1{\cdot}005 Pr^{1/2}$, respectively, see Gebhart [13]. The asymptotic limits, also shown on Fig. 2, are then

$$B = 0{\cdot}461 Pr^{-1/6}, \quad Pr \to \infty$$
$$= 0{\cdot}891 Pr^{1/3}, \quad Pr \to 0.$$

The points taken from stability calculations, for $Pr = 0{\cdot}01$, 0·025, 0·1, 0·733, 2·5 and 6·7, converted through equation (2) to B, are also plotted. Unlike in the uniform flux formulation, the two asymptotes do not as accurately represent the points for $Pr = 0{\cdot}1$–$6{\cdot}7$. For these Prandtl numbers, the $[-\phi'(0)]$ value obtained from base flow solutions and the G^* value used in determination of B^*, are both smaller than the values calculated from the corresponding high Pr asymptotic relationship. Therefore, when converted through equation (2), the B values based on stability calculations lie below high Pr asymptote. See Fig. 2. However, $B = 0{\cdot}315 Pr^{-0{\cdot}065}$ collects all these points within 10 per cent.

COMPARISON WITH EXPERIMENTAL DATA

The data points determined from various investigations are listed in Table 2 and are also plotted in Fig. 2. There is a lot of experimental data in water and in air. Studies of Godaux and Gebhart [8] and Jaluria and Gebhart [10] were concerned with the experimental investigation of the processes during natural transition, from laminar to turbulent flow, of the natural convection flow of water adjacent to a uniform flux vertical surface. The former was a study of thermal transition, while the latter concerned both thermal and velocity transition. Both were a detailed study of the frequency of temperature and velocity disturbances at various x and q'' conditions. In the region of highly amplified disturbances or beginning of transition, the experimental records of fluctuations showed a single predominant frequency. As transition progressed, this characteristic frequency was found to increase. Recent temperature measurements taken by Shaukatullah [14], in the same flow circumstance, in the region of highly amplified disturbances, also demonstrated a single dominant frequency. Jaluria and Gebhart [9], with artificially introduced disturbances, also found the same behavior.

The data points are listed in Table 2, each with the local flow condition indicated. The region around $Pr = 6{\cdot}7$ in Fig. 2 contains all this data taken in water. The spread is ± 8 per cent. These data are shown in detail at the top right-hand corner of Fig. 2.

The results of Polymeropolous and Gebhart [3] correspond to the first appearance of wave-like temperature disturbances in interferograms for flow adjacent to a uniform flux vertical plate in pressurized N_2 ($Pr = 0{\cdot}72$), subject only to naturally occurring disturbances. The B^* value for their results and the B value from the results of Eckert and Soehngen [1], discussed earlier, both lie somewhat above the relevant theoretical curve. The discrepancy is probably due to the insensitivity of the interferometer to small local disturbances and is analogous to the differences noted for transition in supersonic boundary layers between Schlieren measurements and thermocouple measurements, as noted by Schubauer and Klebanoff [15]. That is, waves are first detected by interferometry downstream of the location shown by thermocouple measurements. The above measurements could, therefore, have actually been made in transition, which results in higher frequencies, as noted by Godaux and Gebhart [8] and by Jaluria and Gebhart [10].

The points of Knowles and Gebhart [16], for flow of silicone oil ($Pr = 7{\cdot}7$) over a uniform flux vertical plate, are also shown. The points correspond to hot wire determined bursts.

Recently, an experimental investigation of transition processes in liquid mercury ($Pr = 0{\cdot}025$), for the uniform flux surface condition on surfaces of low thermal capacity, has been reported by Humphreys [17]. This was a channel study. When one of the walls was removed, the flow circumstance corresponded to that adjacent to an isolated vertical plate. Their frequency data, soon to be published, in laminar flow

Table 2. Experimental characteristic frequency data for various fluids

Investigator	Fluid	*Pr*	Flow circumstance	*J*	G^* or G	β^* or β	$9B^*$ or B	Remarks
Knowles and Gebhart [16]	Silicone oil	7·7	Uniform flux	0	745	0·0191	0·521	Appearance of bursts in hot wire output
					620	0·0325	0·809	
					580	0·0380	0·915	
					635	0·0365	0·920	
Jaluria and Gebhart [10]	Water	6·7	Uniform flux	0	600	0·0258	0·634	HA†
					550	0·0277	0·651	BT†
					600	0·0285	0·700	BT
Jaluria and Gebhart [9]	Water	6·7	Uniform flux	0	405	0·0298	0·600	HA for artificially induced disturbance
					530	0·0260	0·600	
					575	0·0254	0·610	
Jaluria and Gebhart [11]	Water	6·7	Uniform flux	1·10	494	0·0366	0·812	HA
					445	0·0391	0·825	HA
				2·16	484	0·0380	0·835	HA
				1·10	371	0·0378	0·73	HA for artificially induced disturbance
					495	0·0344	0·765	
				2·30	371	0·0406	0·785	
Godaux and Gebhart [8]	Water	6·33	Uniform flux	0	890	0·0205	0·605	HT
					508	0·0288	0·649	BT
					878	0·0224	0·663	BT
					600	0·0288	0·705	IT†
					612	0·0303	0·749	IT
Shaukatullah [14]	Water	5·89	Uniform flux	0	680	0·0266	0·694	HA
		5·52	Uniform flux	0	692	0·0270	0·710	HA
					637	0·0250	0·630	HA
					802	0·0258	0·730	HA
		5·15	Uniform flux	0	842	0·0240	0·695	HA
Eckert and Soehngen [1]	Air	0·733	Isothermal	0	400	0·0643	0·414	First appearance of waves in interferogram
Polymeropoulos [3]	N_2	0·72	Uniform flux	0	675	0·0560	1·450	First appearance of waves in interferogram
					704	0·0490	1·300	
					675	0·0510	1·325	
					705	0·0550	1·460	

†HA—Highly amplified disturbances; BT—Beginning of transition; IT—In transition region.

with highly amplified disturbances, will be very close to the low Prandtl number asymptote.

We may conclude that, from observations made in laminar flow with highly amplified disturbances and at the beginning of transition, there is general agreement between predicted filtering and the data. Also, the frequencies corresponding to later stages of transition are greater than the characteristic frequency, in accord with the recent detailed measurements in which it was found that the characteristic frequency increased rapidly downstream during transition.

IN A STRATIFIED AMBIENT MEDIUM

Jaluria and Gebhart [11] also calculated and measured the events leading to transition in a stably and thermally stratified water ambient medium. The surface was uniform flux. From their stability plane, for $Pr = 6{\cdot}7$ and for $J = 1$ and $J = 2$, the values of B^* corresponding $A = 8$ are 0·811 and 0·910, respectively. Here J is the stratification parameter as defined in Table 1. Experiments determined the frequency of the most highly amplified disturbances for both artificially induced and natural disturbances. To preserve clarity, these experimental points are not shown in Fig. 2, but are tabulated. They are in reasonable agreement with the calculated ones given above. Both calculated and measured values of B^* are above the $J = 0$ results, and progressively above for increasing J.

CONCLUSIONS

The characteristic frequency depends only on Prandtl number and heating condition. The agreement between stability predictions and experimental evidence is good. This implies that our curves predict the characteristic frequency in any vertical natural convection flow for both uniform flux and isothermal surface conditions, for any value of Prandtl number.

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FREQUENCE CARACTERISTIQUE DE PERTURBATION DANS L'ECOULEMENT VERTICAL DE CONVECTION NATURELLE

Résumé—De nombreuses expériences, ainsi que des analyses de stabilité, font apparaitre que l'écoulement de couche limite verticale en convection naturelle filtre nettement les perturbations en faveur d'une seule fréquence essentielle (fréquence caractéristique) lorsque ces perturbations sont transportées vers l'aval. L'analyse montre que la fréquence caractéristique, $B^* = \beta^* G^{*1/2}$ ou $B = \beta G^{1/3}$ vers l'aval, dans des conditions respectives d'écoulements à flux constant et isothermes, est fonction seulement du nombre de Prandtl. Les valeurs particulières sont calculées à partir des plans de stabilité pour de nombreuses valeurs du nombre de Prandtl ainsi que pour $Pr \to 0$ et pour $Pr \to \infty$. Les courbes résultantes de B^* et B en fonction du nombre de Prandtl sont en bon accord avec les résultats expérimentaux existants pour des fluides variés et pour des conditions de chauffage et des conditions ambiantes différentes.

CHARAKTERISTISCHE STÖRUNGSFREQUENZ BEI SENKRECHTER FREIER KONVEKTION

Zusammenfassung—Viele Experimente und auch Stabilitätsuntersuchungen zeigen, daß eine senkrechte freie Konvektionsgrenzschicht Störungen im wesentlichen für eine bestimmte Frequenz (charakteristische Frequenz) scharf herausfiltert, wenn die Störungen in Strömungsrichtung wandern. Die Untersuchungen zeigen, daß die charakteristische Freqenz $B^* = \beta^* . G^{*1/2}$ bzw. $B = \beta . G^{1/3}$ sowohl für den Fall konstanter Wärmestromdichte als auch für konstante Wandtemperatur nur eine Funktion der Prandtl–Zahl ist. Die einzelnen Werte sind berechnet für zahlreiche Prandtl–Zahlen sowie für $Pr \to 0$ und $Pr \to \infty$. Die gewonnenen Kurven von B^* und B, aufgetragen über der Prandtl–Zahl, zeigen eine gute Übereinstimmung mit den Versuchsergebnissen für verschiedene Fluide unterschiedlicher Beheizung und Umgebungszustände.

ХАРАКТЕРИСТИЧЕСКАЯ ЧАСТОТА ВОЗМУЩЕНИЙ В ВЕРТИКАЛЬНОМ ПОТОКЕ ПРИ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ

Аннотация — Из многочисленных экспериментов и анализа устойчивости следует, что вертикальный пограничный слой при естественной конвекции фильтрует возмущения с одной частотой (характеристическая частота), так как возмущения перемещяются за счет конвекции вниз по потоку. Анализ указывает, что характеристическая частота B^*—$\beta^* G^{*1/2}$ или $\beta G^{1/3}$ вниз по потоку соответственно для условий постоянного теплового потока или изотермического течения является только функцией числа Прандтля. С помощью карт устойчивости рассчитаны величины для многочисленных чисел Прандтля, при $Pr \to 0$ и $Pr \to \infty$. Существующие экспериментальные данные для различных жидкостей при различных условиях нагрева и окружающей среды коррелируются с помощью результирующих кривых для B^* или B в зависимости от Pr